FIELDS IN NONUNIFORM TRANSMISSION LINES

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Abstract

Baseband waves in nonuniform transmission lines are analyzed by a perturbation procedure emphasizing one-dimensional axial propagation. Successive orders of approximation generate both localized fields and new waves propagating out from nonuniform regions.

Introduction

Nonuniform lines have traditionally been treated by distributed circuit analysis, which gives a heuristic approximation of unknown validity because of incomplete physical content. These equations have validity that goes beyond the simple identification of uniform line solutions with regorous TEM field solutions of simple form.

Field theories for nonuniform lines

Mode conversion problems have stimulated a class of field theories of nonuniform waveguides [2,3,4]. An infinite set of coupled equations is derived for propagation of modes suitably redefined at each cross-section in terms of uniform waveguide modes for that section. If all modes but the dominant mode are neglected the nonuniform line equations are recovered. These results appear to establish the connection between the field equations and distributed circuit theory but do not yield systematic improved approximations short of complete solution, and do not make the role of the taper length scale readily apparent. The modal theory is not useful for even simple questions such as determining minimum length scale of a coax line transition of nominal uniform impedance. The modal theory is a good tool for study of multimode waveguides but gives little physical insight into baseband applications with a single dominant wave species. Something more akin to the "warped mode" concept of Fox[5] is needed.

The key idea is found in the reinterpretation of Solymar's result[6] as a proof that the nonuniform line equation gives the first term in an asymptotic expansion in the taper scale parameter. A perturbation technique of Cole[7] is extended to generate this expansion. This complements the synthesis approach of Mo, Papas, and Baum[8].

The tapered plate line

We use the tapered plate line of Fig.1. as a simple example to illustrate the structure of the theory. The dominant "mode" solution is TM to z with magnetic field satisfying the boundary value problem.

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} - \mu\varepsilon \frac{\partial^2 H_y}{\partial t^2} = 0 \tag{1}$$

with

$$\frac{\partial H_y}{\partial x} = \pm \frac{a_o}{\ell_o}\, a'\left(\frac{z}{\ell_o}\right) \frac{\partial H_y}{\partial z} \text{ on } x = \pm\, a_o\, a\left(\frac{z}{\ell_o}\right) \tag{2}$$

Introduce normalized coordinates $x^\dagger = x/a_o$, $z^\dagger = z/\ell_o$, and $t^\dagger = t/((\mu\varepsilon)^{\frac{1}{2}}\ell_o)$, where the time scale is normalized to the transit time through the section. The dimensionless ratio $\eta = a_o/\ell_o$ is used as the perturbation expansion parameter in the limit $\eta \to 0$. With these normalized variables, η makes an explicit appearance as

$$\frac{\partial^2 H_y}{\partial x^{\dagger 2}} + \eta^2 \left\{ \frac{\partial^2 H_y}{\partial z^{\dagger 2}} - \frac{\partial^2 H_y}{\partial t^{\dagger 2}} \right\} = 0 \tag{3}$$

with

$$\frac{\partial H_y}{\partial x^\dagger} = \eta^2\, a'(z^\dagger) \frac{\partial H_y}{\partial z^\dagger} \text{ on } x^\dagger = \pm\, a(z^\dagger) \tag{4}$$

The limit $\eta \to 0$ may be regarded as either $a_o \to 0$ or $\ell_o \to \infty$. The former emphasizes directly the dominance of transverse quasistatic behaviour, while the latter shows the behaviour for typical cross-section as the taper is made more gradual with corresponding lengthening of time scale. In (4) the factor η^2 couples transverse and longitudinal effects which remain uncoupled in the TEM solution for uniform line. Expand H_y in a power series in η^2.

$$H_y(x,z,t) = H_o(x^\dagger,z^\dagger,t^\dagger) + \eta^2\, H_2(x^\dagger,z^\dagger,t^\dagger) + .. \tag{4}$$

and substitute in the equations. After classification by powers of η^2 an ordered sequence of problems is obtained. In order $O(\eta^o)$

$$\frac{\partial^2 H_o}{\partial x^{\dagger 2}} = 0 \text{ with } \frac{\partial H_o}{\partial x^\dagger} = 0 \text{ on } x^\dagger = \pm\, a(z^\dagger) \tag{6}$$

This has solution

$$H_o(x^\dagger,z^\dagger,t^\dagger) = A_o(z^\dagger,t^\dagger) \tag{7}$$

where A_o is an as yet arbitrary function of $z^\dagger$ and $t^\dagger$ only and constant with $x^\dagger$. This is just the form of solution envisaged in distributed circuit theory. Higher order problems have the form in order $O(\eta^{2n})$

$$\frac{\partial^2 H_{2n}}{\partial x^{\dagger 2}} = - \left\{ \frac{\partial^2}{\partial x^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}} \right\} H_{2n-2} \tag{8}$$

with

$$\frac{\partial H_{2n}}{\partial x^\dagger} = \pm\, a'(z^\dagger) \frac{\partial H_{2n-2}}{\partial z^\dagger} \text{ on } x^\dagger = \pm\, a(z^\dagger) \tag{9}$$

Substitute solution (6) for H_o and integrate the $O(\eta^2)$ problem once to obtain

$$\frac{\partial^2 A_o}{\partial x^{\dagger 2}} + \frac{a'(z^\dagger)}{a(z^\dagger)} \frac{\partial A_o}{\partial z^\dagger} - \frac{\partial^2 A_o}{\partial t^{\dagger 2}} = 0 \tag{10}$$

which is when denormalized identical to the nonuniform line equation for this structure. Distributed circuit theory stops here, but the perturbation analysis now generates correction terms of order $O(\eta^2)$ and higher. A further integration yields:

$$H_2 = -\frac{x^{\dagger 2}}{2}\left\{\frac{\partial^2}{\partial z^{\dagger 2}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\} A_o + A_2\ (z^\dagger, t^\dagger) \qquad (11)$$

where $A_2(z^\dagger, t^\dagger)$ is a new arbitrary function to be determined from consideration of the $O(\eta^4)$ problem. The first term depends only on the lower order solution and vanishes outside the taper region, and A_2 is found to satisfy an inhomogeneous version of the nonuniform line equation (10) with source terms depending on the taper profile and lower order solution. This sets the general pattern for higher order solutions, with localized field distortion terms, and new propagating terms which will be apparent as in TDR measurements outside the tapered section. An exact solution in the frequency domain is available for a uniform wedge and an expansion in η^2 yields term by term agreement with the perturbation series carried to $O(\eta^6)$.

Curved centerlines

The perturbation analysis may be extended to nonuniform tapered plate lines defined symmetrically on a curved centerline, Fig.2, with the line profile defined on the normals of the centerline defined by $x = x_o(\xi)$, $z = z_o(\zeta)$, where ζ is arclength along the centerline. Cartesian coordinates of a point in this system are given by

$$x(\xi,\zeta) = x_o(\zeta) + \xi \cos\Psi(\zeta) \qquad (12)$$

$$z(\xi,\zeta) = z_o(\zeta) - \xi \sin\Psi(\zeta) \qquad (13)$$

This orthogonal coordinate system is uniquely defined for our problem provided successive normals do not intersect within the transmission line region. The condition for this intersection is $\xi_o\ d\Psi = d\zeta$, so the boundary profiles must satisfy $b(\zeta) \lessgtr \xi_o$. The differential equation for the transverse magnetic field becomes

$$\frac{1}{(1-\xi\Psi')}\ \frac{\partial}{\partial\xi}\{(1-\xi\Psi')\ \frac{\partial H}{\partial\xi}\} +$$

$$\frac{1}{(1-\xi\Psi')}\ \frac{\partial}{\partial\zeta}\{\frac{1}{1-\xi\Psi'}\ \frac{\partial H}{\partial\zeta}\} - \mu\varepsilon\ \frac{\partial^2}{\partial t^2} = 0 \qquad (14)$$

with boundary condition for perfectly conducting walls

$$\frac{\partial H}{\partial\xi} = \frac{\pm b'(\zeta)}{(1-\xi\Psi'(\zeta))^2}\ \frac{\partial H}{\partial\zeta}$$

$$\text{on } \xi = \pm b(\zeta) = \pm a_o\ a(\zeta/\ell_o) \qquad (15)$$

If the scaling procedure is applied to these equations by introducing normalized variables $\xi^\dagger = \xi/a_o$ and $\xi^\dagger = \zeta/\ell_o$ with the functional form of the angle Ψ rewritten as $\psi(\zeta^\dagger)$. The scaling procedure with $\eta = a_o/\ell_o \to 0$ can be thought of as either a shrinking of the transverse dimension a_o on a given centerline, or as a stretching of the centerline scale ℓ_o in which geometrical similarity is preserved. In normalized variables the governing equations are:

$$\frac{\partial^2 H}{\partial\xi^{\dagger 2}} - \frac{\eta\psi'}{(1-\eta\xi^\dagger\psi')}\ \frac{\partial H}{\partial\xi^\dagger} + \frac{\eta^2}{(1-\eta\xi^\dagger\psi')^2}\ \circ$$

$$\frac{\partial^2 H}{\partial\xi^{\dagger 2}} + \frac{\eta^3\xi^\dagger\psi''}{(1-\eta\xi^\dagger\psi')^3}\ \frac{\partial H}{\partial\zeta^\dagger} - \frac{\partial^2 H}{\partial t^{\dagger 2}} = 0 \qquad (17)$$

with

$$\frac{\partial H}{\partial\xi^\dagger} = \frac{\pm\eta^2 a'(\zeta^\dagger)}{(1-\eta\xi^\dagger\psi')^2}\ \frac{\partial H}{\partial\zeta^\dagger} \text{ on } \xi^\dagger = \pm a(\zeta^\dagger) \qquad (18)$$

Given coordinate uniqueness the denominator terms may be expanded in convergent power series, and odd powers of η will occur in both the differential equation and boundary condition.

The perturbation expansion for H will now need to contain odd order terms

$$H = H_o + \eta H_1 + \eta^2 H_2 + \eta^3 H_3 + \ldots \qquad (16)$$

The standard procedure yields immediately $H_o = A_o(\zeta^\dagger, t^\dagger)$ and $H_1 = A_1(\zeta^\dagger, t^\dagger)$ as arbitrary functions of $\zeta^\dagger$ and $t^\dagger$ only. The equation in $O(\eta^2)$ for H_2 does not contain any curvature terms and the solution is formally similar to the straight centerline analysis, determining A_o and associated field distortion terms, and introducing a further undetermined propagator $A_2(\zeta^\dagger, t^\dagger)$. The curvature makes its first permanent appearance in the third order equation. Detailed analysis[1] here shows that the first order undetermined propagator A_1 satisfies the same equation as A_o. Since no new information is added we may take $H_1 = 0$. The solution for H_3 contains localized field distortion terms proportional to centerline curvature. Symmetry arguments show that all odd order new propagators vanish, so that the odd order terms contribute to the fields only within the nonuniform section.

The first curvature related terms visible outside the nonuniform region are obtained in determining $A_2(\zeta^\dagger, t^\dagger)$ from partial solution of the sub-problem for H_4 as curvature dependent source terms for the nonuniform line equation for A_2. Thus in the complete solution of the curved tapered plate line problem the dominant curvature dependent effects are of second order in the common parameter for taper and curvature scale.

A particular case of some interest is the curved uniform line where all orders of propagating waves obey the simple wave equation. The solution for A_2 yields the one-dimensional wave equation,

$$\frac{\partial^2 A_2}{\partial\zeta^{\dagger 2}} - \frac{\partial^2 A_2}{\partial t^{\dagger 2}} = -\frac{a^2\psi'^2}{3}\left\{\frac{\partial^2 H_o}{\partial\zeta^{\dagger 2}} + \frac{2\psi''}{\psi'}\ \frac{\partial H_o}{\partial\zeta^\dagger}\right\} \qquad (19)$$

The source terms are independent of the sign of the curvature which fits the physical intuition that a reflectometer measurement should not be able to distinguish a given bend from a similar bend in the opposite sense.

Consider a curved uniform line with sufficiently smooth variation of curvature from an initial straight section as a model for a time domain reflectometer experiment. For unit step function excitation the second order reflected wave observed in the initial straight section, with time measured from the incident step at the same position.

$$A_2(t^\dagger) = -\frac{a^2}{3}\ \{\frac{d}{dx}\ \psi(\frac{x}{2})\}^2_{x = t^\dagger} \qquad (20)$$

which is proportional to the square of the curvature at a round-trip travel time $t^{\dagger}$ down the line.

Coaxial Lines

Circular coaxial lines, besides being of great technical importance, provide a good three dimensional application for our analysis in the sense that convenient laboratory realizations require relatively little idealization to obtain a tractable theoretical model. Also a grossly nonuniform coaxial line with straight centerline can have constant impedance in distributed circuit analysis.

The perturbation analysis for nonuniform coaxial line on a straight centerline may be set up in exactly similar fashion to the tapered plate line analysis, with the expected variations for rotational symmetry[1]. A similar structure of solution exists with propagating terms in even order given by inhomogeneous one dimensional nonuniform line equations with the differential operator identical to that derived from distributed circuit theory. Escalation of complexity with order is even more rapid.

Propagation in low impedance coaxial line, Fig.3. is to physical intuition dominated by the presence of the walls rather than the distant centerline. Reflect for example on the situation where the inner and outer walls are both in radial planes, or even curving backwards. A longitudinal section through such a geometry is reminiscent of the tapered plate line with curved centerline of the previous section. A suitable orthogonal coordinate system is obtained by rotation of that two dimensional coordinate system about the coaxial line axis. This yields a coordinate system centered in the propagation region as shown in Fig.3. If ϕ is the azimuthal angle, then with radius ρ_o substituted for x_o in (12,13) the Cartesian coordinates (x,y,z) of a point in the (ξ,ζ,ϕ) system are

$$z = z_o(\zeta) - \xi \sin\Psi(\zeta) \tag{21}$$

$$x = (\rho_o(\zeta) + \xi \cos\Psi(\zeta)) \cos\phi \tag{22}$$

$$y = (\rho_o(\zeta) + \xi \cos\Psi(\zeta)) \sin\phi \tag{23}$$

It is convenient to let the mean radius scale with ℓ_o, although a scaling intermediate between a_o and ℓ_o would preserve the low impedance character. The perturbation analysis follows the pattern of the curved normal centerline analysis but with increased complexity. The basic nonuniform transmission line equation which results is

$$\left\{\frac{\partial^2}{\partial\zeta^{\dagger 2}} + \left(\frac{a'}{a} - \frac{\sin\psi}{\rho_o^{\dagger}}\right)\frac{\partial}{\partial\zeta^{\dagger}} - \frac{\partial^2}{\partial t^{\dagger 2}}\right\}(\rho H_\phi) = 0 \tag{24}$$

where H_ϕ is the physical azimuthal magnetic field component. A physical interpretation of this equation is not difficult to find, recalling that $\sin\Psi = d\rho_o/d\zeta$. Since mean circumference is proportional to ρ_o, this term measures the logarithmic rate of change of circumference. If the low impedance line is viewed as a finite width tapered plate line curled up transversely then the first derivative term in (24) expresses total impedance variation due to width and spacing changes. In this version edge effects are automatically taken care of. The simple appearance of (24) is deceptive since a great deal of structural information has been built into the coordinate system used to express the equation. For example it contains the exact TEM solution for a radial transmission line between parallel plates, obtained by setting a = const. and $\Psi = \pi/2$.

Conclusion

Perturbation analysis of the field theory of nonuniform lines shows that distributed circuit theory is the basic approximation of a systematic sequence involving the taper and curvature scale parameter. The approach is well suited to obtaining powerful but formally simple approximate descriptions in some grossly nonuniform situations, by suitable structuring of the coordinate system used.

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References

1. This work is based on part of a thesis submitted in partial fulfillment of the Ph.D. requirements in Electrical Engineering and Physics at the California Institute of Technology, 1972. See also Caltech Antenna Lab. Report No. 60, AFOSR-TR-72-0198 October, 1971.

2. S.A. Schelkunoff, "Conversion of Maxwell's equations into generalized telegraphists' equations", Bell Syst. Tech. J., 34, pp.995-1044 September, 1955.

3. B.Z. Katzenelenbaum, "Nonuniform waveguides with slowly changing parameters", Dokl. Akad. Nauk. SSSR, 12, pp.711-714, 1955.

4. G. Reiter, "Generalized telegraphists' equation for waveguides of varying cross-section", Proc. IEEE (London), 106B, Suppl. 13, pp.54-57, 1959.

5. A. G. Fox, "Wave coupling by warped normal modes", Bell Syst. Tech. J., 34, pp.823-852, July, 1955.

6. L. Solymar, "Spurious mode generation in nonuniform waveguide", IRE Trans. on Microwave Theory and Techniques, MTT-7 pp.379-383, July, 1959.

7. J. D. Cole, "Perturbation methods in applied mathematics", Blaisdell, Waltham MA, 1968, Ch. 4.5.3.

8. T.C. Mo, C.H. Papas, C.E. Baum, "General scaling method for electromagnetic fields with application to a matching problem", J. Math. Phys., 14, pp. 479-483, April, 1973.

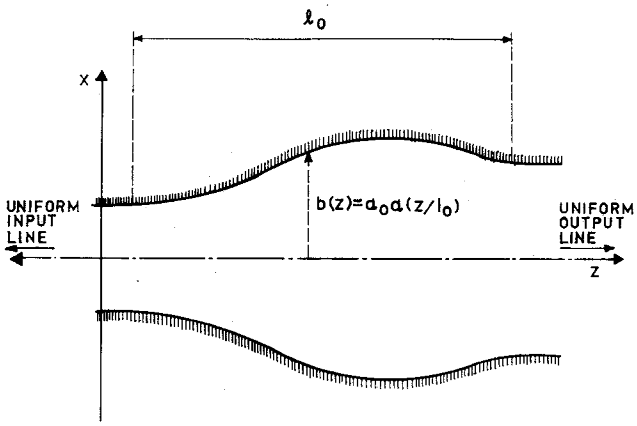

Fig. 1 Geometry for symmetrical tapered plate line

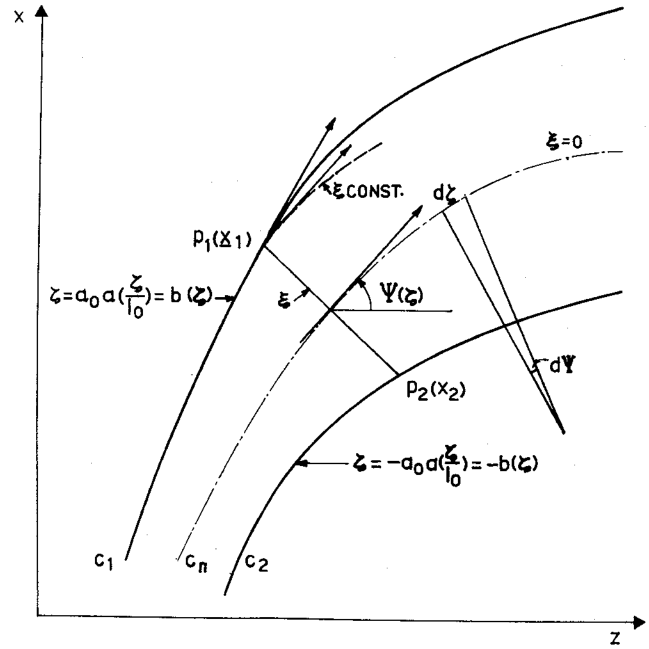

Fig. 2 Geometry for tapered plate line defined on the normals to a curved centerline

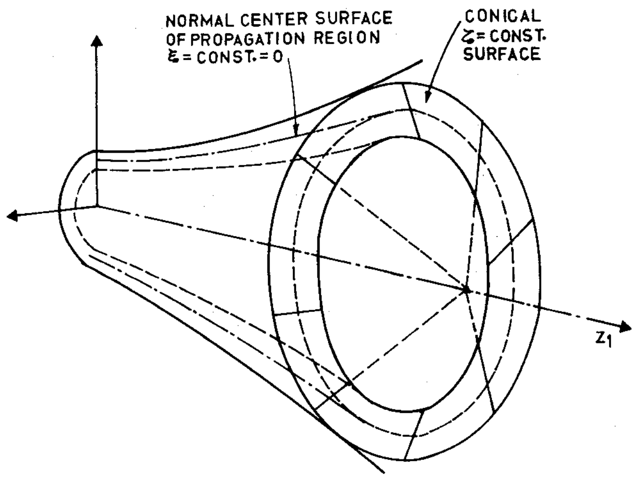

Fig. 3 Conical coordinate system for propagation region of low impedance nonuniform coaxial line.